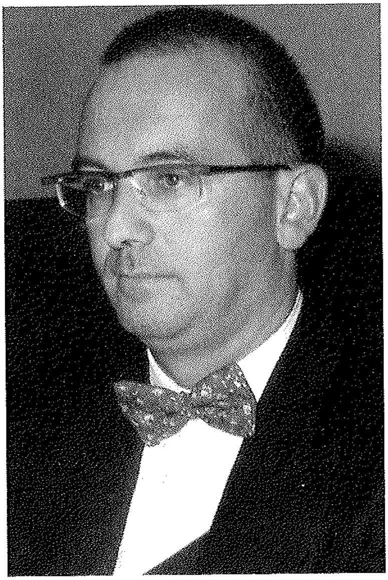
Stochastic Programming: Resolving Uncertainty with Barycentric Approximation



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Dear Dr. Latsis,
Prof. Thorens,
Ladies and Gentlemen,

It is a great honour for me to have the opportunity to present the key parts of my work under the title "Stochastic Programming: Resolving Uncertainty with Barycentric Approximation". I would like to express my deepest thanks to the Latsis family and their Foundation.

My presentation intends to provide insights into the basic ideas of stochastic programming (a field of activity within *mathematical programming)* paying particular attention to the methodological concept of barycentric approximation.

Introduction

The field of *mathematical programming* was mainly initiated by the mathematician G. B. Dantzig in the mid'40s. Dantzig focused on the ability to state objectives and then find optimal policy solutions to pratical decision problems of great complexity. Additionally, he considered the relations between the set of items being consumed or produced and the set of associated activities or production processes, which led to the incorporation of constraints in decision problems. Practitioners usually refer to their various plans and proposed schedules as *programs.* This was the motivation for G. B. Dantzig to introduce the notion *programming* in 1947. The terms *linear programming* and *mathematical programming* were introduced by the economists T. C. Koopmans in 1948 and R. Dorfman in 1949.

Dantzig was motivated to generalize the steady-state Input-Output Model of W. Leontief to a dynamic model, one that could change over time. He had in mind *planning dynamically over time,* particularly *planning under uncertainty*. This way, *stochastic programming* began to emerge as one important part of *mathematical programming*. The complexity of interaction between time and uncertainty makes practical decision and planning problems to utmost difficult applications of probability and optimization theory. *Stochastic programming* combines these two fields with the intention to design sophisticated, analytical tools for analysing interaction effects between decision making and uncertainty. *Barycentric approximation* may be seen as one of such tools. It will be presented in light of an actual problem: the optimal funding of variable rate mortgages.

A practical problem statement

Within the last decades, the asset and liability structures of banks changed considerably. Due to growing institutional savings, the retail savings volume has decreased. This development caused banks to fund a considerable part of variable rate mortgage volume with money and capital market instruments.

Funding variable rate mortgages with bonds of different maturities is a planning problem, which requires periodic decision making based on the current structure of maturing funds, the stochastic evolvement of variable rate mortgages, and the interest rate curve. This planning problem may be modelled and solved via *stochastic programming:* monthly, the institution has to decide on its funding strategy; how much of the current mortgage volume is to be borrowed with maturities of 1, 2, 3, 6 months, 1, 2, 3, 5, 7 or 10 years? The objective is to minimize the expected funding costs, taking into account the risk the institution is exposed to by the uncertain evolvement of mortgage volume and interest rates over a predefined planning horizon.

For the ease of exposition, we focus on a one-month period. The uncertain evolvement of interest rate and mortgage volume within the next month may be visualized with a set of points (Figure 1).

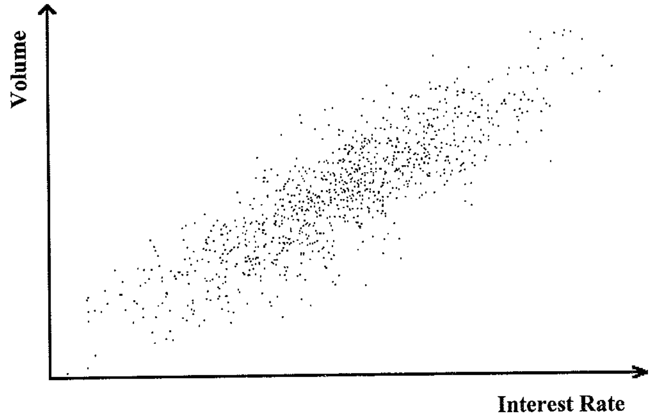

Figure 1: Empirical data (market movements)

These points may, e.g., represent empirical data of the past. It is only for illustrative purposes that the interest rate is drawn in one dimension. The positive correlation of these data refer to the dependency between interest rate level and the volume of variable rate mortgages: low interest rates induce a small volume of variable rate mortgages, as clients prefer fixed rate mortgages, high interest rates motivate clients to prefer variable rate mortgages.

Covering market movements and pricing decisions

The first step to resolve uncertainty is covering the probabilistic events (i.e. the market data) by a box with a sufficient confidence level. The box stands for a Cartesian product of two simplices; in our case, working with a one-dimensional mortgage volume and a one-dimensional interest rate means that we have the Cartesian product of two intervals forming the box (Figure 2). There are two edges AB and CD that allow for varying the interest rates and two edges AD and BC that allow for barying the mortgage volume.

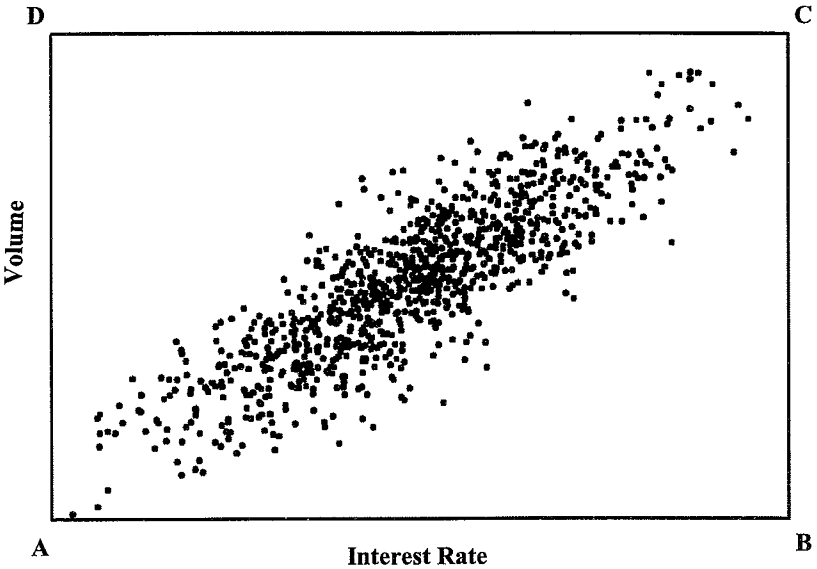

Figure 2: Covering stochastic market movements

Today, at the beginning of the planning period, our aim is to find the optimal funding strategy based on the stochastic evolvement of the market data interest rate and mortgage volume up to next months. For any funding strategy of today one can determine its value in one month based on the then observed interest rate and mortgage volume. In other words, we may determine the future value of today's funding strategy given a specific combination of future market data (interest rate and mortgage volume). The function which shows the cost of a specific funding strategy dependent on market data is denoted *value function* or *cost function* (Figure 3). Clearly, different funding strategies of today yield different value functions. it is natural to choose the best strategy with the lowest expected costs. It has to be stressed that the value function is not given analytically, even not explicitly, but implicitly by the opti-

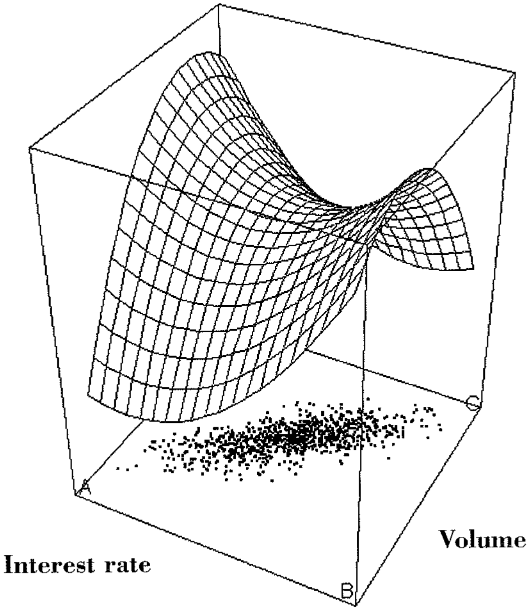

Figure 3: Value function: pricing a decision

mal solutions of a stochastic multistage optimization problem. The numerical effort for evaluating only a single value of the implicitly given cost function takes about 1 hour even on very powerful workstations. Hence, one cannot practically afford to evaluate the cost functions for hundreds of points, as this would take days until a single funding strategy is priced, without having performed any optimization procedure. Therefore, it becomes a necessity to rely on a sufficiently accurate approximation of these value functions. This approximation represents the second step to resolving uncertainty.

Upper and lower approximations of saddle functions

As mentioned before, the value functions are given implicitly by the optimal solutions of a stochastic multistage optimization problem. In

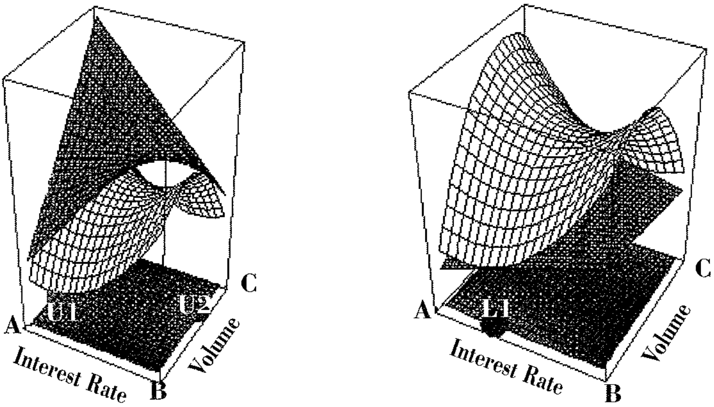

Figure 4: Bilinear approximation of the value function

case that the underlying stochastic multistage optimization problem is convex (and this, fortunately, happens in many practical situations), applying the *convex duality theory* due to Fenchel-Rockafeller-Moreau, the value function takes the shape of a saddle function. More precisely, the value function has a positive curvature (a convex shape) with respect to the interest rates and a negative curvature (a concave shape) with respect to the mortgage volume. It is appropriate to design an approximation technique which exploits the inherent saddle structure of the value functions. This can be motivated in a rather easy way: take two points U1 and U2, one on each boundary, which allow for varying the mortgage volume. Take the corresponding linear (first-order) approximation of the cost function with respect to the mortgage volume and form a bilinear upper approximation of the cost function over the entire box ABCD, which is supposed to cover the stochastic market movements sufficiently by the end of next period. Analogously, take tow points L1 and L2, one on each boundary, which allow for varying the interest rate. Take the corresponding linear (first-order) approximation of the cost function with respect to the interest rate and form a bilinear lower approximation of the cost function over the same box ABCD. These lower and upper bilinear approximations are not only easy to integrate, as only expectations and covariances of mortgage volume and interest rates are needed, but also allow for stating and localizing the current inaccuracies, so that potential improvements can be achieved via successive box partitionings.

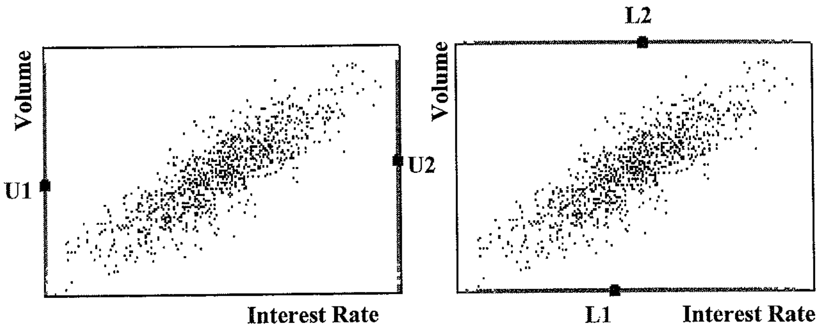

Figure 5: Discretization of stochastic market movements

Having the goal to minimize the inaccuracy in mind, it can be observed that the bilinear approximations (and hence the level of inaccuracy) vary with the pair of points (U1, U2) and (L1, L2). It remains to be answered which pairs are optimal in the sens that they yield the best approximation of the expected costs. Those who are familiar with mathematical optimization realize that this amounts to the optimal solution of a so-called semi-infinite optimization problem. It is the saddle structure which allows that this question can be solved analytically with explicit formulae. This is the third and last step to resolving uncertainty.

Discretization of stochastic movements

The derivation of the numerical result can be visualized in the following way: Imagine that these points (which refer to the market data mortgage volume and interest rates) represent a mass distribution with a mass equal to 1. Perform a projection of these points onto the volume boundaries so that the distance of the mass points to the boundaries is taken into account. This decomposes the entire unit mass into two component parts. Each boundary obtains its own mass distribution; the mass on each boundary is less than one, but together, they add up to the unity mass. Since now the volume boundaries represent simplices equipped with a particular mass, one can evaluate the associated generalized barycenters U1, U2. It can be proven that these two barycenters U1, U2 represent the optimal points where the value function should be supported from above for minimizing the expected error. Observe that these barycenters differ in the volume coordinate; this asymmetry is caused by the positive correlation of the two stochastic components, mortgage volume and interest rates.

The same argumentation is applicable for the derivation of the best lower approximation. Now, the projection of market data is performed to the interest rate boundaries (which are simplices in the multidimensional case), taking into account the distance of the points to those boundaries, obtaining again two mass distributions which determine the corresponding barycenters L1 and L2. These represent the optimal points for supporting the value function from below. These barycenters are asymmetric with respect to the mortgage volume because of the positive correlation.

This way, two discrete distributions are obtained which are optimal solutions of the so-called generalized moment problems, the dual formulations of the semi-infinite programs mentioned earlier. In exactly this sense, these discrete approximations may be viewed as the best approximations of the stochastic market movements, being completely determined by the expectations and covariances of interest rate and mortgage volume. Based on these barycenters, that are small in number even after some box refinements, the funding strategies can be priced sufficiently fast so that the optimization can be performed.

Conclusions and perspectives

This approach has been successfully applied to the problem of funding variable rate mortgages in cooperation with a Swiss bank. Practitioners have accepted that the barycenters may be interpreted as a very important bundle of market scenarios, based on which the funding strategies should be priced and optimized. It is certainly encouraging that up to now a promising performance can be reported. However, it must be stressed that there is no guarantee for the future.

Today, I have focused on a particular methodology which has the potential to solve complex models. In the presentation almost nothing has been revealed concerning modelling details. Bus those details are the key elements for mapping the real problem with sufficient goodness and, in the sequel, for preserving the achieved performance.

Within the methodological as well as the modelling component, numerous challenging issues that have arised will attract further scientific research activities.

Ladies and gentlemen, at this stage, I would like to thank Prof. Kall, head of the Institute of Operations Research at the University of Zurich, and Prof. Stähly, head of the Institute of Operations Research at the University of St. Gallen. Both supported and strongly stimulated this work, provided me with an excellent work environment and made these developments possible.

Expressing my deep gratitude the Latsis Foundation, I would like to thank you, Ladies and Gentlemen, for your attention.

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